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**Some new inequalities
of Hermite–Hadamard type
for GA -convex functions**

ABSTRACT. Some new inequalities of Hermite–Hadamard type for GA -convex functions defined on positive intervals are given. Refinements and weighted version of known inequalities are provided. Some applications for special means are also obtained.

1. Introduction. Let $J \subset (0, \infty)$ be an interval; a real-valued function $f : J \rightarrow \mathbb{R}$ is said to be GA -convex (concave) on J (see for instance [2]), if

$$(1.1) \quad f\left(x^{1-\lambda}y^\lambda\right) \leq (\geq) (1-\lambda)f(x) + \lambda f(y)$$

for all $x, y \in J$ and $\lambda \in [0, 1]$.

Since the condition (1.1) can be written as

$$(1.2) \quad \begin{aligned} & f \circ \exp((1-\lambda)\ln x + \lambda\ln y) \\ & \leq (\geq) (1-\lambda)f \circ \exp(\ln x) + \lambda f \circ \exp(\ln y), \end{aligned}$$

then we observe that $f : J \rightarrow \mathbb{R}$ is GA -convex (concave) on J if and only if $f \circ \exp$ is convex (concave) on $\ln J := \{\ln z, z \in J\}$. If $J = [a, b]$, then $\ln J = [\ln a, \ln b]$.

It is known that the function $f(x) = \ln(1+x)$ is GA -convex on $(0, \infty)$ [2].

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For real and positive values of x , the *Euler gamma* function Γ and its *logarithmic derivative* ψ , the so-called *digamma function*, are defined by

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t} dt \text{ and } \psi(x) := \frac{\Gamma'(x)}{\Gamma(x)}.$$

It has been shown in [28] that the function $f : (0, \infty) \rightarrow \mathbb{R}$ defined by

$$f(x) = \psi(x) + \frac{1}{2x}$$

is *GA*-concave on $(0, \infty)$ while the function $g : (0, \infty) \rightarrow \mathbb{R}$ defined by

$$g(x) = \psi(x) + \frac{1}{2x} + \frac{1}{12x^2}$$

is *GA*-convex on $(0, \infty)$.

Example 1. If $[a, b] \subset (0, \infty)$ and the function $g : [\ln a, \ln b] \rightarrow \mathbb{R}$ is convex (concave) on $[\ln a, \ln b]$, then the function $f : [a, b] \rightarrow \mathbb{R}$, $f(t) = g(\ln t)$ is *GA*-convex (concave) on $[a, b]$.

We recall that the classical Hermite–Hadamard inequality states that

$$(1.3) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t) dt \leq \frac{f(a) + f(b)}{2}$$

for any convex function $f : [a, b] \rightarrow \mathbb{R}$.

For related results, see [1]–[6], [8]–[11], [12]–[18] and [20]–[27].

The *identric mean* $I(a, b)$ is defined by

$$I(a, b) := \frac{1}{e} \left(\frac{b^b}{a^a} \right)^{\frac{1}{b-a}}$$

while the *logarithmic mean* is defined by

$$L(a, b) := \frac{b-a}{\ln b - \ln a}$$

where $0 < a < b$.

In [28] the authors obtained the following Hermite–Hadamard type inequality.

Theorem 1. If $b > a > 0$ and $f : [a, b] \rightarrow \mathbb{R}$ is a differentiable *GA*-convex (concave) function on $[a, b]$, then

$$(1.4) \quad \begin{aligned} f(I(a, b)) &\leq (\geq) \frac{1}{b-a} \int_a^b f(t) dt \\ &\leq (\geq) \frac{b-L(a, b)}{b-a} f(b) + \frac{L(a, b)-a}{b-a} f(a). \end{aligned}$$

The differentiability of the function is not necessary in Theorem 1 for the first inequality (1.4) to hold, as shown in [10].

If we take $\lambda = \frac{1}{2}$ in the definition (1.1) of GA -convex (concave) function on $[a, b]$, then we have

$$(1.5) \quad f(\sqrt{ab}) \leq (\geq) \frac{f(a) + f(b)}{2}.$$

The following refinement of (1.5), which is an inequality of Hermite–Hadamard type, holds (see [25] for an extension for GA h -convex functions):

Theorem 2. *Let $f : [a, b] \subset (0, \infty) \rightarrow \mathbb{R}$ be a GA -convex (concave) function on $[a, b]$. Then we have*

$$(1.6) \quad f(\sqrt{ab}) \leq (\geq) \frac{1}{\ln b - \ln a} \int_a^b \frac{f(t)}{t} dt \leq (\geq) \frac{f(a) + f(b)}{2}.$$

Motivated by the above results we provide in the following a refinement of (1.6) for a division of the interval $[a, b]$. We also establish a weighted version that generalizes the inequalities (1.4) and (1.6) and provides upper and lower bounds for the moments

$$\frac{1}{b-a} \int_a^b t^p f(t) dt, \text{ for } p \in \mathbb{R} \text{ with } p \neq 0, -1$$

of the GA -convex (concave) function on $[a, b]$.

2. Some refinements. In 1994, [5] (see also [17, p. 22]) we proved the following refinement of Hermite–Hadamard inequality. For the sake of completeness we give here a direct proof that is different from the one in [5].

Lemma 1. *Let $g : [c, d] \rightarrow \mathbb{R}$ be a convex function on $[c, d]$. Then for any division $c = x_0 < x_1 < \dots < x_{n-1} < x_n = d$ with $n \geq 1$ we have the inequalities*

$$(2.1) \quad \begin{aligned} g\left(\frac{c+d}{2}\right) &\leq \frac{1}{d-c} \sum_{i=0}^{n-1} (x_{i+1} - x_i) g\left(\frac{x_{i+1} + x_i}{2}\right) \\ &\leq \frac{1}{d-c} \int_c^d g(x) dx \\ &\leq \frac{1}{d-c} \sum_{i=0}^{n-1} (x_{i+1} - x_i) \frac{g(x_i) + g(x_{i+1})}{2} \\ &\leq \frac{1}{2} [g(c) + g(d)]. \end{aligned}$$

Proof. Using the Hermite–Hadamard inequality on the interval $[x_i, x_{i+1}]$, $i \in \{0, \dots, n-1\}$, we have

$$(2.2) \quad \begin{aligned} (x_{i+1} - x_i) g\left(\frac{x_{i+1} + x_i}{2}\right) &\leq \int_{x_i}^{x_{i+1}} g(x) dx \\ &\leq (x_{i+1} - x_i) \frac{g(x_i) + g(x_{i+1})}{2} \end{aligned}$$

for any $i \in \{0, \dots, n-1\}$.

Summing in (2.2) over $i \in \{0, \dots, n-1\}$ and dividing by $d - c$, we get the second and the third inequalities in (2.1).

Since for $p_i := \frac{x_{i+1} - x_i}{d - c} \geq 0$ we have $\sum_{i=0}^{n-1} p_i = 1$, then by Jensen's inequality we have

$$\begin{aligned} \frac{1}{d - c} \sum_{i=0}^{n-1} (x_{i+1} - x_i) g\left(\frac{x_{i+1} + x_i}{2}\right) &\geq g\left(\frac{1}{d - c} \sum_{i=0}^{n-1} (x_{i+1} - x_i) \left(\frac{x_{i+1} + x_i}{2}\right)\right) \\ &= g\left(\frac{1}{d - c} \sum_{i=0}^{n-1} \left(\frac{x_{i+1}^2 - x_i^2}{2}\right)\right) \\ &= g\left(\frac{d^2 - c^2}{2(d - c)}\right) = g\left(\frac{c + d}{2}\right), \end{aligned}$$

which proves the first inequality in (2.1).

For a convex function $g : [c, d] \rightarrow \mathbb{R}$ we have

$$g(x) = g\left(\frac{(d - x)c + (x - c)d}{d - c}\right) \leq \frac{(d - x)g(c) + (x - c)g(d)}{d - c}$$

for any $x \in [c, d]$.

This implies that

$$g(x_i) \leq \frac{1}{d - c} [(d - x_i)g(c) + (x_i - c)g(d)]$$

and

$$g(x_{i+1}) \leq \frac{1}{d - c} [(d - x_{i+1})g(c) + (x_{i+1} - c)g(d)]$$

for any $i \in \{0, \dots, n-1\}$.

Therefore,

$$\begin{aligned} &\sum_{i=0}^{n-1} (x_{i+1} - x_i) \frac{g(x_i) + g(x_{i+1})}{2} \\ &\leq \frac{1}{2(d - c)} \sum_{i=0}^{n-1} (x_{i+1} - x_i) [(d - x_i)g(c) \\ &\quad + (x_i - c)g(d) + (d - x_{i+1})g(c) + (x_{i+1} - c)g(d)] \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{d-c} \left[\left(d(d-c) - \frac{1}{2} (d^2 - c^2) \right) g(c) + \left(\frac{1}{2} (d^2 - c^2) - c(d-c) \right) g(d) \right] \\
&= (d-c) \frac{g(c) + g(d)}{2},
\end{aligned}$$

and the last part of (2.1) is proved. \square

Theorem 3. Let $f : [a, b] \subset (0, \infty) \rightarrow \mathbb{R}$ be a GA -convex function on $[a, b]$. Then for any division $a = t_0 < t_1 < \dots < t_{n-1} < t_n = b$ with $n \geq 1$ we have the inequalities

$$\begin{aligned}
(2.3) \quad f(\sqrt{ab}) &\leq \frac{1}{\ln b - \ln a} \sum_{i=0}^{n-1} (\ln t_{i+1} - \ln t_i) f(\sqrt{t_i t_{i+1}}) \\
&\leq \frac{1}{\ln b - \ln a} \int_a^b \frac{f(s)}{s} ds \\
&\leq \frac{1}{\ln b - \ln a} \sum_{i=0}^{n-1} (\ln t_{i+1} - \ln t_i) \frac{f(t_i) + f(t_{i+1})}{2} \\
&\leq \frac{1}{2} [f(a) + f(b)].
\end{aligned}$$

Proof. If $f : [a, b] \subset (0, \infty) \rightarrow \mathbb{R}$ is a GA -convex function on $[a, b]$, then $f \circ \exp$ is convex on $[\ln a, \ln b]$. Let $c = \ln a$, $d = \ln b$ and the division $x_i = \ln t_i$, $i \in \{0, \dots, n-1\}$ of the interval $[c, d]$.

If we write the inequality (2.1) for $g = f \circ \exp$ on the interval $[c, d]$ and for the division

$$\ln a = \ln t_0 < \ln t_1 < \dots < \ln t_{n-1} < \ln t_n = \ln b,$$

we have

$$\begin{aligned}
(2.4) \quad &(f \circ \exp) \left(\frac{\ln a + \ln b}{2} \right) \\
&\leq \frac{1}{\ln b - \ln a} \sum_{i=0}^{n-1} (\ln t_{i+1} - \ln t_i) (f \circ \exp) \left(\frac{\ln t_{i+1} + \ln t_i}{2} \right) \\
&\leq \frac{1}{\ln b - \ln a} \int_{\ln a}^{\ln b} (f \circ \exp)(x) dx \\
&\leq \frac{1}{\ln b - \ln a} \sum_{i=0}^{n-1} (\ln t_{i+1} - \ln t_i) \frac{(f \circ \exp)(\ln t_i) + (f \circ \exp)(\ln t_{i+1})}{2} \\
&\leq \frac{1}{2} [(f \circ \exp)(\ln a) + (f \circ \exp)(\ln b)],
\end{aligned}$$

that is equivalent to

$$\begin{aligned}
 f(\sqrt{ab}) &\leq \frac{1}{\ln b - \ln a} \sum_{i=0}^{n-1} (\ln t_{i+1} - \ln t_i) f(\sqrt{t_i t_{i+1}}) \\
 &\leq \frac{1}{\ln b - \ln a} \int_{\ln a}^{\ln b} (f \circ \exp)(x) dx \\
 (2.5) \quad &\leq \frac{1}{\ln b - \ln a} \sum_{i=0}^{n-1} (\ln t_{i+1} - \ln t_i) \frac{f(t_i) + f(t_{i+1})}{2} \\
 &\leq \frac{1}{2} [f(a) + f(b)].
 \end{aligned}$$

By using the change of variable $\exp(x) = t$, we have $x = \ln t$, $dx = \frac{dt}{t}$ and

$$\int_{\ln a}^{\ln b} (f \circ \exp)(x) dx = \int_a^b \frac{f(t)}{t} dt$$

and by (2.5) we get the desired result (2.3). \square

Corollary 1. Let $f : [a, b] \subset (0, \infty) \rightarrow \mathbb{R}$ be a GA -convex function on $[a, b]$. If $a \leq t \leq b$, then we have

$$\begin{aligned}
 f(\sqrt{ab}) &\leq \frac{1}{\ln b - \ln a} [(\ln t - \ln a) f(\sqrt{at}) + (\ln b - \ln t) f(\sqrt{tb})] \\
 (2.6) \quad &\leq \frac{1}{\ln b - \ln a} \int_a^b \frac{f(s)}{s} ds \\
 &\leq \frac{1}{2} \left[f(t) + \frac{(\ln t - \ln a) f(a) + (\ln b - \ln t) f(b)}{\ln b - \ln a} \right] \\
 &\leq \frac{1}{2} [f(a) + f(b)].
 \end{aligned}$$

Remark 1. Let $f : [a, b] \subset (0, \infty) \rightarrow \mathbb{R}$ be a GA -convex function on $[a, b]$. For a division $0 = \lambda_0 < \lambda_1 < \dots < \lambda_{n-1} < \lambda_n = 1$ with $n \geq 1$ of the interval $[0, 1]$, consider the division

$$(2.7) \quad t_i := a \left(\frac{b}{a} \right)^{\lambda_i} = a^{1-\lambda_i} b^{\lambda_i}, \quad i \in \{0, \dots, n-1\}$$

of the interval $[a, b]$.

Observe that

$$\begin{aligned}
 \ln t_{i+1} - \ln t_i &= \ln \left[a \left(\frac{b}{a} \right)^{\lambda_{i+1}} \right] - \ln \left[a \left(\frac{b}{a} \right)^{\lambda_i} \right] \\
 &= \ln a + \lambda_{i+1} \ln \left(\frac{b}{a} \right) - \ln a - \lambda_i \ln \left(\frac{b}{a} \right) \\
 &= (\ln b - \ln a) (\lambda_{i+1} - \lambda_i)
 \end{aligned}$$

and

$$\sqrt{t_i t_{i+1}} = \sqrt{a^{1-\lambda_i} b^{\lambda_i} a^{1-\lambda_{i+1}} b^{\lambda_{i+1}}} = a^{1-\frac{\lambda_i+\lambda_{i+1}}{2}} b^{\frac{\lambda_i+\lambda_{i+1}}{2}}$$

for any $i \in \{0, \dots, n-1\}$.

If we write the inequality (2.3) for the division (2.7) we get

$$\begin{aligned} f(\sqrt{ab}) &\leq \sum_{i=0}^{n-1} (\lambda_{i+1} - \lambda_i) f\left(a^{1-\frac{\lambda_i+\lambda_{i+1}}{2}} b^{\frac{\lambda_i+\lambda_{i+1}}{2}}\right) \\ &\leq \frac{1}{\ln b - \ln a} \int_a^b \frac{f(s)}{s} ds \\ (2.8) \quad &\leq \sum_{i=0}^{n-1} (\lambda_{i+1} - \lambda_i) \frac{f(a^{1-\lambda_i} b^{\lambda_i}) + f(a^{1-\lambda_{i+1}} b^{\lambda_{i+1}})}{2} \\ &\leq \frac{1}{2} [f(a) + f(b)], \end{aligned}$$

for any division $0 = \lambda_0 < \lambda_1 < \dots < \lambda_{n-1} < \lambda_n = 1$ with $n \geq 1$ of the interval $[0, 1]$.

If we write the inequality (2.8) for $0 < \lambda < 1$, then we get

$$\begin{aligned} f(\sqrt{ab}) &\leq (1-\lambda) f\left(a^{\frac{1-\lambda}{2}} b^{\frac{\lambda+1}{2}}\right) + \lambda f\left(a^{\frac{2-\lambda}{2}} b^{\frac{\lambda}{2}}\right) \\ (2.9) \quad &\leq \frac{1}{\ln b - \ln a} \int_a^b \frac{f(t)}{t} dt \\ &\leq \frac{1}{2} \left[f\left(a^{1-\lambda} b^{\lambda}\right) + (1-\lambda) f(b) + \lambda f(a) \right] \\ &\leq \frac{f(a) + f(b)}{2}. \end{aligned}$$

The inequality (2.9) was obtained in [10].

In the following section we establish some weighted Hermite–Hadamard type inequalities for GA -convex functions.

3. Weighted inequalities. We have the following weighted inequality:

Theorem 4. Let $f : [a, b] \subset (0, \infty) \rightarrow \mathbb{R}$ be a GA -convex function on $[a, b]$ and $w : [a, b] \rightarrow [0, \infty)$ an integrable function on $[a, b]$ with $\int_a^b w(t) dt > 0$, then

$$\begin{aligned} (3.1) \quad f\left(\exp\left(\frac{\int_a^b w(t) \ln t dt}{\int_a^b w(t) dt}\right)\right) &\leq \frac{\int_a^b w(t) f(t) dt}{\int_a^b w(t) dt} \\ &\leq \frac{\left(\ln b - \frac{\int_a^b w(t) \ln t dt}{\int_a^b w(t) dt}\right) f(a) + \left(\frac{\int_a^b w(t) \ln t dt}{\int_a^b w(t) dt} - \ln a\right) f(b)}{\ln b - \ln a}. \end{aligned}$$

Proof. Observe that for $t \in [a, b]$ we have

$$\ln t = \frac{(\ln b - \ln t) \ln a + (\ln t - \ln a) \ln b}{\ln b - \ln a}.$$

By the convexity of $f \circ \exp$ we have

$$\begin{aligned} f(t) &= (f \circ \exp)(\ln t) \\ (3.2) \quad &= (f \circ \exp) \left(\frac{(\ln b - \ln t) \ln a + (\ln t - \ln a) \ln b}{\ln b - \ln a} \right) \\ &\leq \frac{\ln b - \ln t}{\ln b - \ln a} f(a) + \frac{\ln t - \ln a}{\ln b - \ln a} f(b) \end{aligned}$$

for any $t \in [a, b]$.

If we multiply (3.2) by $w(t) \geq 0$, $t \in [a, b]$ and integrate over t on $[a, b]$, then we get

$$\begin{aligned} &\int_a^b f(t) w(t) dt \\ &\leq \frac{\ln b \int_a^b w(t) dt - \int_a^b w(t) \ln t dt}{\ln b - \ln a} f(a) + \frac{\int_a^b w(t) \ln t dt - \ln a \int_a^b w(t) dt}{\ln b - \ln a} f(b) \end{aligned}$$

and the second inequality in (3.1) is proved.

By Jensen's inequality we have

$$\frac{\int_a^b w(t) f(t) dt}{\int_a^b w(t) dt} = \frac{\int_a^b w(t) (f \circ \exp)(\ln t) dt}{\int_a^b w(t) dt} \geq (f \circ \exp) \left(\frac{\int_a^b w(t) \ln t dt}{\int_a^b w(t) dt} \right)$$

and the first part of (3.1) is proved. \square

Corollary 2 (see Theorem 2). *Let $f : [a, b] \subset (0, \infty) \rightarrow \mathbb{R}$ be a GA-convex function on $[a, b]$, then*

$$(3.3) \quad f(\sqrt{ab}) \leq \frac{1}{\ln b - \ln a} \int_a^b \frac{f(t)}{t} dt \leq \frac{f(a) + f(b)}{2}.$$

Proof. If we take $w(t) = \frac{1}{t}$ in (3.1), then we have

$$\begin{aligned} (3.4) \quad &f \left(\exp \left(\frac{\int_a^b \frac{\ln t}{t} dt}{\int_a^b \frac{1}{t} dt} \right) \right) \leq \frac{\int_a^b \frac{f(t)}{t} dt}{\int_a^b \frac{1}{t} dt} \\ &\leq \frac{\left(\ln b - \frac{\int_a^b \frac{\ln t}{t} dt}{\int_a^b \frac{1}{t} dt} \right) f(a) + \left(\frac{\int_a^b \frac{\ln t}{t} dt}{\int_a^b \frac{1}{t} dt} - \ln a \right) f(b)}{\ln b - \ln a}. \end{aligned}$$

Since

$$\int_a^b \frac{\ln t}{t} dt = \frac{1}{2} \left[(\ln b)^2 - (\ln a)^2 \right], \quad \int_a^b \frac{1}{t} dt = \ln b - \ln a,$$

then we get from (3.4)

$$\begin{aligned} f\left(\exp\left(\frac{\ln b + \ln a}{2}\right)\right) &\leq \frac{\int_a^b \frac{f(t)}{t} dt}{\ln b - \ln a} \\ &\leq \frac{(\ln b - \frac{\ln b + \ln a}{2}) f(a) + (\frac{\ln b + \ln a}{2} - \ln a) f(b)}{\ln b - \ln a} \end{aligned}$$

and the inequality (3.3) is proved. \square

Define the p -logarithmic mean as

$$L_p(a, b) := \left(\frac{1}{p+1} \frac{b^{p+1} - a^{p+1}}{b - a} \right)^{1/p}, \quad p \neq -1, 0$$

for $0 < a < b$.

Corollary 3. Let $f : [a, b] \subset (0, \infty) \rightarrow \mathbb{R}$ be a GA-convex function on $[a, b]$, then for any $p \in \mathbb{R}$ with $p \neq 0, -1$ we have

$$\begin{aligned} (3.5) \quad &L_p^p(a, b) f\left([I(a^{p+1}, b^{p+1})]^{\frac{1}{p+1}}\right) \\ &\leq \frac{1}{b-a} \int_a^b t^p f(t) dt \\ &\leq \frac{(L(a^{p+1}, b^{p+1}) - a^{p+1}) f(a) + (b^{p+1} - L(a^{p+1}, b^{p+1})) f(b)}{(p+1)(b-a)}. \end{aligned}$$

If $p = 0$, then we have

$$\begin{aligned} (3.6) \quad &f(I(a, b)) \leq \frac{1}{b-a} \int_a^b f(t) dt \\ &\leq \frac{(L(a, b) - a) f(a) + (b - L(a, b)) f(b)}{b-a}. \end{aligned}$$

Proof. If we take $w(t) = t^p$, with $p \neq 0, -1$ in (3.1), then we have

$$\begin{aligned} (3.7) \quad &f\left(\exp\left(\frac{\int_a^b t^p \ln t dt}{\int_a^b t^p dt}\right)\right) \\ &\leq \frac{\int_a^b t^p f(t) dt}{\int_a^b t^p dt} \\ &\leq \frac{\left(\ln b - \frac{\int_a^b t^p \ln t dt}{\int_a^b t^p dt}\right) f(a) + \left(\frac{\int_a^b t^p \ln t dt}{\int_a^b t^p dt} - \ln a\right) f(b)}{\ln b - \ln a}. \end{aligned}$$

We have

$$\int_a^b t^p dt = \frac{1}{p+1} (b^{p+1} - a^{p+1})$$

and

$$\begin{aligned}
 \int_a^b t^p \ln t dt &= \frac{1}{p+1} \int_a^b \ln t dt^{p+1} \\
 &= \frac{1}{p+1} \left[t^{p+1} \ln t \Big|_a^b - \int_a^b t^p dt \right] \\
 &= \frac{1}{p+1} \left(b^{p+1} \ln b - a^{p+1} \ln a - \frac{1}{p+1} (b^{p+1} - a^{p+1}) \right).
 \end{aligned}$$

Then

$$\begin{aligned}
 \frac{\int_a^b t^p \ln t dt}{\int_a^b t^p dt} &= \frac{\frac{1}{p+1} \left(b^{p+1} \ln b - a^{p+1} \ln a - \frac{1}{p+1} (b^{p+1} - a^{p+1}) \right)}{\frac{1}{p+1} (b^{p+1} - a^{p+1})} \\
 &= \frac{1}{p+1} \ln I(a^{p+1}, b^{p+1}) = \ln [I(a^{p+1}, b^{p+1})]^{\frac{1}{p+1}},
 \end{aligned}$$

$$\begin{aligned}
 \ln b - \frac{\int_a^b t^p \ln t dt}{\int_a^b t^p dt} &= \ln b - \frac{1}{p+1} \left[\frac{b^{p+1} \ln b^{p+1} - a^{p+1} \ln a^{p+1}}{b^{p+1} - a^{p+1}} - 1 \right] \\
 &= \frac{1}{p+1} \left(1 - \frac{\ln b^{p+1} - \ln a^{p+1}}{b^{p+1} - a^{p+1}} a^{p+1} \right) = \frac{1}{p+1} \left(1 - \frac{a^{p+1}}{L(a^{p+1}, b^{p+1})} \right) \\
 &= \frac{1}{p+1} \frac{L(a^{p+1}, b^{p+1}) - a^{p+1}}{L(a^{p+1}, b^{p+1})}
 \end{aligned}$$

and, similarly,

$$\frac{\int_a^b t^p \ln t dt}{\int_a^b t^p dt} - \ln a = \frac{1}{p+1} \frac{b^{p+1} - L(a^{p+1}, b^{p+1})}{L(a^{p+1}, b^{p+1})}.$$

From (3.7) we then have

$$\begin{aligned}
 &f \left(\exp \left(\ln [I(a^{p+1}, b^{p+1})]^{\frac{1}{p+1}} \right) \right) \\
 &\leq \frac{\frac{1}{b-a} \int_a^b t^p f(t) dt}{\frac{1}{b-a} \int_a^b t^p dt} \\
 &\leq \frac{\left(\frac{1}{p+1} \frac{L(a^{p+1}, b^{p+1}) - a^{p+1}}{L(a^{p+1}, b^{p+1})} \right) f(a) + \left(\frac{1}{p+1} \frac{b^{p+1} - L(a^{p+1}, b^{p+1})}{L(a^{p+1}, b^{p+1})} \right) f(b)}{\ln b - \ln a},
 \end{aligned}$$

i.e.

$$\begin{aligned}
 & f \left(\left[I \left(a^{p+1}, b^{p+1} \right) \right]^{\frac{1}{p+1}} \right) \\
 & \leq \frac{\frac{1}{b-a} \int_a^b t^p f(t) dt}{L_p^p(a, b)} \\
 & \leq \frac{\left(\frac{1}{p+1} \frac{L(a^{p+1}, b^{p+1}) - a^{p+1}}{L(a^{p+1}, b^{p+1})} \right) f(a) + \left(\frac{1}{p+1} \frac{b^{p+1} - L(a^{p+1}, b^{p+1})}{L(a^{p+1}, b^{p+1})} \right) f(b)}{\ln b - \ln a} \\
 & = \frac{\frac{L(a^{p+1}, b^{p+1}) - a^{p+1}}{L(a^{p+1}, b^{p+1})} f(a) + \frac{b^{p+1} - L(a^{p+1}, b^{p+1})}{L(a^{p+1}, b^{p+1})} f(b)}{\ln b^{p+1} - \ln a^{p+1}}.
 \end{aligned}$$

By multiplying this inequality with $L_p^p(a, b)$, we get

$$\begin{aligned}
 & L_p^p(a, b) f \left(\left[I \left(a^{p+1}, b^{p+1} \right) \right]^{\frac{1}{p+1}} \right) \\
 & \leq \frac{1}{b-a} \int_a^b t^p f(t) dt \\
 & \leq \frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \frac{\frac{L(a^{p+1}, b^{p+1}) - a^{p+1}}{L(a^{p+1}, b^{p+1})} f(a) + \frac{b^{p+1} - L(a^{p+1}, b^{p+1})}{L(a^{p+1}, b^{p+1})} f(b)}{\ln b^{p+1} - \ln a^{p+1}} \\
 & = \frac{(L(a^{p+1}, b^{p+1}) - a^{p+1}) f(a) + (b^{p+1} - L(a^{p+1}, b^{p+1})) f(b)}{(p+1)(b-a)}.
 \end{aligned}$$

If we perform the calculations in the above inequalities for $p = 0$, we get the desired inequality (3.6). We omit the details. \square

Remark 2. If we take $p = 1$ in (3.5), then we get

$$\begin{aligned}
 (3.8) \quad & f \left(\sqrt{I(a^2, b^2)} \right) \leq \frac{1}{b-a} \int_a^b t f(t) dt \\
 & \leq \frac{(A(a, b) L(a, b) - a^2) f(a) + (b^2 - A(a, b) L(a, b)) f(b)}{2(b-a)}.
 \end{aligned}$$

4. Applications. Let $q \neq 0$ and consider the convex function $g(t) = \exp(qt)$, $t \in \mathbb{R}$. Then the function $f_q : (0, \infty) \rightarrow \mathbb{R}$, $f_q(t) = g(\ln t) = \exp(q \ln t) = t^q$ is a GA -convex function on $(0, \infty)$. We observe that for $0 < a < b$ we have

$$\begin{aligned}
 \frac{1}{b-a} \int_a^b t^q dt &= \begin{cases} \frac{1}{q+1} \frac{b^{q+1} - a^{q+1}}{b-a}, & q \neq -1 \\ \frac{\ln b - \ln a}{b-a}, & q = -1 \end{cases} \\
 &= \begin{cases} L_q^q(a, b), & q \neq -1 \\ L^{-1}(a, b), & q = -1, \end{cases}
 \end{aligned}$$

where $L_q(a, b)$ ($q \neq -1$) is the q -Logarithmic mean and L is the logarithmic mean defined in the introduction.

If we write the inequality (2.8) for the GA -convex function on $(0, \infty)$, $f_q : (0, \infty) \rightarrow \mathbb{R}$, $f_q(t) = t^q$, $q \neq 0$, then we get the following result:

Proposition 1. *For any division $0 = \lambda_0 < \lambda_1 < \dots < \lambda_{n-1} < \lambda_n = 1$ with $n \geq 1$ of the interval $[0, 1]$,*

$$\begin{aligned}
 G^q(a, b) &\leq \sum_{i=0}^{n-1} (\lambda_{i+1} - \lambda_i) G^q(a^{2-\lambda_i-\lambda_{i+1}}, b^{\lambda_i+\lambda_{i+1}}) \\
 (4.1) \quad &\leq L(a, b) L_{q-1}^{q-1}(a, b) \\
 &\leq \sum_{i=0}^{n-1} (\lambda_{i+1} - \lambda_i) \frac{G^q(a^{2(1-\lambda_i)}, b^{2\lambda_i}) + G^q(a^{2(1-\lambda_{i+1})}, b^{2\lambda_{i+1}})}{2} \\
 &\leq A(a^q, b^q).
 \end{aligned}$$

If we write the inequality (3.5) for the GA -convex function on $(0, \infty)$, $f_q : (0, \infty) \rightarrow \mathbb{R}$, $f_q(t) = t^q$, $q \neq 0$, then:

Proposition 2. *If $0 < a < b$ and $p, q \in \mathbb{R}$ with $p \neq 0, -1$, $q \neq 0$, then*

$$\begin{aligned}
 (4.2) \quad &L_p^p(a, b) [I(a^{p+1}, b^{p+1})]^{\frac{q}{p+1}} \\
 &\leq L_{p+q}^{p+q}(a, b) \\
 &\leq \frac{(L(a^{p+1}, b^{p+1}) - a^{p+1}) a^q + (b^{p+1} - L(a^{p+1}, b^{p+1})) b^q}{(p+1)(b-a)}.
 \end{aligned}$$

From (3.6) and (3.8) we get:

Proposition 3. *If $0 < a < b$ and $q \in \mathbb{R}$ with $q \neq 0$, then*

$$(4.3) \quad I^q(a, b) \leq L_q^q(a, b) \leq \frac{(L(a, b) - a) a^q + (b - L(a, b)) b^q}{b - a},$$

and

$$\begin{aligned}
 (4.4) \quad &I^{q/2}(a^2, b^2) \leq L_{q+1}^{q+1}(a, b) \\
 &\leq \frac{(A(a, b) L(a, b) - a^2) a^q + (b^2 - A(a, b) L(a, b)) b^q}{2(b-a)}.
 \end{aligned}$$

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